JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 25, No. 1, January–February 2002

Rendezvous Equations in a Central-Force Field with Linear Drag

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The terminal phase of a rendezvous between a satellite and a spacecraft in a central-force field with a drag force that is linear in the velocity is considered. To simplify the work, we assume identical drag coefficients on both. In this general setting, we linearize the equations of motion of the spacecraft and show that they can be transformed into a reasonably simple form. Under certain simplifying assumptions, they can be reduced to one second-order linear differential equation. We then specialize to the case of inverse square laws. If we assume that the drag accelerations are roughly identical on the satellite and spacecraft, we obtain a set of linear differential equations that can be solved in terms of integrals. This enables us to represent the solution of this version of the problem in terms of a state-transition matrix. The work is then placed in the context of various control models that have been developed in previous work. The kind of transformations presented may be useful in the analysis of problems having more realistic drag models.

Introduction

THE rendezvous of a spacecraft with a satellite in a Newtonian gravitational field has many obvious and practical applications. One of the earliest researchers in this area was Lawden.[1] The terminal phase of the rendezvous has been studied using linear models.[2–10] In many applications of terminal rendezvous studies, the linearized equations can be reduced effectively to one second-order equation, simplifying the analysis. Linearity and the application of optimization techniques then determine the optimal rendezvous trajectory, in principle. Further generalization of these results to the terminal rendezvous problem in a general central-force field[11] shows that the same reduction to one second-order equation can be accomplished in this more general context.

The present paper builds on previous work[11–15] by analyzing the terminal rendezvous problem in a general central force field with linear drag. It is shown that reductions similar to those previously obtained[8,11] apply in this more general context.

The plan of the paper is as follows. We derive some general results about the motion of a particle in a central force field with linear drag. We then treat the rendezvous problem in this setting. Next we specialize to gravitational and drag forces that are inversely proportional to the square of the distance from the center of the attracting body. We then present simplifying assumptions under which the equations of motion can be integrated to define a state-transition matrix of the same form as in previous work.[15] Finally we address the control problems for the linearized equations and show how they fit into the context of previous models.[16–22] We end with some observations and conclusions.

This work is a first step by the authors in an analytical investigation of terminal rendezvous in the presence of drag. A second study, now under way, attempts to apply a similar approach to the more difficult and practical problem of quadratic drag. Future results on the problem of terminal rendezvous with quadratic drag are expected.

Received 22 May 2000; revision received 7 March 2001; accepted for publication 18 May 2001.

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Motion of a Particle in a Central-Force Field with Linear Drag

The equations of motion for the Kepler problem, namely, motion of a particle in the gravitational field of a central body, with linear drag have been considered in the literature by several authors.[12–14] In particular Leach[14] simplified the derivation of some previous results. In this section, we extend these results to a general central-force field. In this context, we derive a new differential identity for the orbit, which will be of paramount importance in reducing the equations for the rendezvous problem.

The equation of motion for a particle in a general central-force field with linear drag is

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - g(\alpha, R)\dot{\boldsymbol{R}}, \qquad R = |\boldsymbol{R}| \tag{1}$$

where f is continuous, $\boldsymbol{R}$ is the radius vector, and dots denote differentiation with respect to time. Taking the vector product of Eq. (1) with $\boldsymbol{R}$ from the left and introducing the angular momentum of the motion

$$\boldsymbol{L} = \boldsymbol{R} \times \dot{\boldsymbol{R}} \tag{2}$$

we obtain

$$\dot{\boldsymbol{L}} + g(\alpha, R)\boldsymbol{L} = 0 \tag{3}$$

hence,

$$\dot{\boldsymbol{L}} \times \boldsymbol{L} = 0 \tag{4}$$

We infer, therefore, that $\dot{\boldsymbol{L}}$ is always parallel to $\boldsymbol{L}$, consequently, $\boldsymbol{L}$ has a constant direction. It follows that the motion is always in a fixed plane. Introducing polar coordinates in this plane to represent the motion, we have

$$\boldsymbol{L} = R^2\dot{\theta}\boldsymbol{\ell} \tag{5}$$

where $\boldsymbol{\ell}$ is a unit vector in the fixed direction of $\boldsymbol{L}$. Substituting Eq. (5) in Eq. (3) and integrating, we obtain

$$L + \int g[\alpha, R(\theta)]R^2(\theta)\, d\theta = h, \qquad |\boldsymbol{L}| = L \tag{6}$$

where h is a constant. To obtain a second conserved quantity (a generalization of Hamilton's vector) we rewrite Eq. (3) as

$$g(\alpha, R) = -\dot{L}/L \tag{7}$$

Substituting Eq. (7) in Eq. (1) and dividing by L leads to

$$\frac{d}{dt}\left(\frac{\dot{\boldsymbol{R}}}{L}\right)+\frac{f(R)\boldsymbol{R}}{L}=0 \tag{8}$$

Using Eq. (5) to substitute for R in Eq. (8) yields

$$\frac{d}{dt}\left(\frac{\dot{\boldsymbol{R}}}{L}\right)+\frac{f\left[(L/\dot{\theta})^{\frac{1}{2}}\right]\boldsymbol{e}_r}{(L\dot{\theta})^{\frac{1}{2}}}=0, \qquad \boldsymbol{e}_r=\frac{\boldsymbol{R}}{R} \tag{9}$$

hence,

$$\boldsymbol{K}=\frac{\dot{\boldsymbol{R}}}{L}+\int\frac{f\left[(L/\dot{\theta})^{\frac{1}{2}}\right]}{(L\dot{\theta})^{\frac{1}{2}}}\boldsymbol{e}_r\,dt \tag{10}$$

is constant, that is, $\boldsymbol{K}$ is a conserved vector. Finally the Runge–Lenz vector for this motion is given by

$$\boldsymbol{J}=\boldsymbol{K}\times\boldsymbol{\ell} \tag{11}$$

We now return to the equations of motion. Because the motion is in a fixed plane, we can introduce a standard radial coordinate system with unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$. The equations of motion along $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_r$, respectively, become

$$R\ddot{\theta}+2\dot{R}\dot{\theta}=-g(\alpha,R)R\dot{\theta} \tag{12}$$

$$\ddot{R}-R\dot{\theta}^2=-f(R)R-g(\alpha,R)\dot{R} \tag{13}$$

From Eq. (12) we obtain

$$R^2\dot{\theta}=h-I(\theta) \tag{14}$$

where

$$I(\theta)=\int g[\alpha,R(\theta)]R^2(\theta)\,d\theta \tag{15}$$

This is a restatement of Eq. (6). We can rewrite Eq. (14) as

$$\frac{d}{dt}=\frac{h-I(\theta)}{R^2}\frac{d}{d\theta} \tag{16}$$

and use this to substitute for d/dt in Eq. (13); after some algebra we then obtain the orbit equation

$$\frac{R''}{R}-2\left(\frac{R'}{R}\right)^2+\frac{f(R)R^4}{[h-I(\theta)]^2}=1 \tag{17}$$

where primes denote differentiation with respect to θ.

Rendezvous Problems

The system we wish to consider consists of a satellite in orbit and a spacecraft in a central force field. The position of the satellite with respect to the field source is given by $\boldsymbol{R}(t)$ and its equation of motion is Eq. (1). The position of the spacecraft measured from the satellite is $\boldsymbol{r}(t)$ and its equation of motion is

$$\ddot{\boldsymbol{R}}+\ddot{\boldsymbol{r}}=-f(|\boldsymbol{R}+\boldsymbol{r}|)(\boldsymbol{R}+\boldsymbol{r})-g(\alpha,|\boldsymbol{R}+\boldsymbol{r}|)(\dot{\boldsymbol{R}}+\dot{\boldsymbol{r}})+\boldsymbol{T}/m \tag{18}$$

where $\boldsymbol{T}$ is the thrust and m is the mass of the spacecraft.

To linearize this problem we introduce some assumptions on f and g: 1) f and g are analytic with $df/dR\neq 0$ and $dg/dR\neq 0$ and 2) $r\ll R$.

Using these assumptions, we now have

$$f(|\boldsymbol{R}+\boldsymbol{r}|)=f\left\{[(\boldsymbol{R}+\boldsymbol{r})\cdot(\boldsymbol{R}+\boldsymbol{r})]^{\frac{1}{2}}\right\}$$

$$=f(R)+\frac{df}{dR}(R)\left(\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}+\frac{r^2}{2R}\right)+\cdots \tag{19}$$

and similarly

$$g(\alpha,|\boldsymbol{R}+\boldsymbol{r}|)=g(\alpha,R)+\frac{dg}{dR}(\alpha,R)\left(\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}+\frac{r^2}{2R}\right)+\cdots \tag{20}$$

Substituting Eqs. (19) and (20) in Eq. (18), using Eq. (1), and linearizing, we obtain

$$\ddot{\boldsymbol{r}}=-f(R)\boldsymbol{r}-\frac{df}{dR}(R)\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}\boldsymbol{R}-g(\alpha,R)\dot{\boldsymbol{r}}$$

$$-\frac{dg}{dR}(\alpha,R)\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}\dot{\boldsymbol{R}}+\frac{\boldsymbol{T}}{m} \tag{21}$$

In a coordinate system rotating with the satellite, Eq. (21) becomes

$$\ddot{\boldsymbol{r}}+2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})+(\dot{\boldsymbol{\Omega}}\times\boldsymbol{r})=-f(R)\boldsymbol{r}$$

$$-\frac{df}{dR}(R)\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}\boldsymbol{R}-g(\alpha,R)(\dot{\boldsymbol{r}}+\boldsymbol{\Omega}\times\boldsymbol{r})$$

$$-\frac{dg}{dR}(\alpha,R)\frac{\boldsymbol{R}\cdot\boldsymbol{r}}{R}(\dot{\boldsymbol{R}}+\boldsymbol{\Omega}\times\boldsymbol{R})+\frac{\boldsymbol{T}}{m} \tag{22}$$

where $\boldsymbol{\Omega}$ is the orbital angular velocity of the satellite.

However, because the motion of the satellite is planar, we can choose the coordinate system attached to it so that the x axis is tangential but opposed to the motion of the satellite, the y axis is in the direction of $\boldsymbol{R}$, and the z axis completes a right-handed system. In this frame, $\boldsymbol{r}=(x,y,z)$, $\boldsymbol{R}=R(0,1,0)$, and $\boldsymbol{\Omega}=(0,0,\dot{\theta})=(0,0,\omega)$.

When this setting is used, Eq. (22) becomes

$$(\ddot{x},\ddot{y},\ddot{z})+2(-\omega\dot{y},\omega\dot{x},0)+(-\omega^2x,-\omega^2y,0)+(-\dot{\omega}y,\dot{\omega}x,0)$$

$$=-f(R)(x,y,z)-\frac{df}{dR}(R)R(0,y,0)$$

$$-g(\alpha,R)[(\dot{x},\dot{y},\dot{z})+(-\omega y,\omega x,0)]$$

$$-\frac{dg}{dR}(\alpha,R)y(-R\omega,\dot{R},0)+\left(\frac{T_1}{m},\frac{T_2}{m},\frac{T_3}{m}\right) \tag{23}$$

Using Eq. (12) we have

$$\dot{R}=(-R/2)[g(\alpha,R)+\dot{\omega}/\omega] \tag{24}$$

Substituting Eq. (24) in Eq. (23) and using Eq. (16) to change variables from t to θ, we obtain in component form

$$\omega^2x''+\omega\omega'x'=[\omega^2-v(\theta)]x+[\omega\omega'+\omega p(\theta)+2\omega u_1(\theta)]y$$

$$+2\omega^2y'-p(\theta)\omega x'+T_1/m \tag{25}$$

$$\omega^2y''+\omega\omega'y'=\left[\omega^2-v(\theta)-q(\theta)+\omega'u_1(\theta)+u_2(\theta)\right]y$$

$$-p(\theta)\omega y'-[\omega\omega'+\omega p(\theta)]x-2\omega^2x'+T_2/m \tag{26}$$

$$\omega^2z''+\omega\omega'z'=-v(\theta)z-p(\theta)\omega z'+T_3/m \tag{27}$$

where primes denote differentiation with respect to θ and

$$v(\theta)=f[R(\theta)], \qquad q(\theta)=\frac{df}{dR}[R(\theta)]R(\theta)$$

$$p(\theta)=g[\alpha,R(\theta)], \qquad u_1(\theta)=\frac{1}{2}\frac{dg}{dR}[\alpha,R(\theta)]R(\theta)$$

$$u_2(\theta)=g[\alpha,R(\theta)]u_1(\theta) \tag{28}$$

To simplify these equations, we now perform the transformation

$$(\bar{x},\bar{y},\bar{z})=\left\{\omega^{\frac{1}{2}}\exp\left[\frac{1}{2}\int\frac{p(\theta)}{\omega(\theta)}d\theta\right]\right\}(x,y,z) \tag{29}$$

Substituting this in Eqs. (25–27) and dividing by

$$B(\theta)=\omega^{\frac{3}{2}}\exp\left[-\frac{1}{2}\int\frac{p(\theta)}{\omega(\theta)}d\theta\right] \tag{30}$$

we obtain the following equations after dropping the overbars:

$$x''+[A(\theta,\omega,\omega',\omega'')-1]x-2y'$$

$$-2\omega^{-1}u_1(\theta)y=T_1/mB(\theta) \tag{31}$$

$$y''+\left\{A(\theta,\omega,\omega',\omega'')-1+[q(\theta)-\omega'u_1(\theta)\right.$$

$$\left.-u_2(\theta)]\big/\omega^2\right\}y+2x'=T_2/mB(\theta) \tag{32}$$

$$z''+A(\theta,\omega,\omega',\omega'')z=T_3/mB(\theta) \tag{33}$$

where

$$A(\theta, \omega, \omega', \omega'') = -\tfrac{1}{2}(\omega''/\omega) + \tfrac{1}{4}(\omega'/\omega)^2 + v(\theta)/\omega^2 - \tfrac{1}{2}[p'(\theta)/\omega] - \tfrac{1}{4}[p^2(\theta)/\omega^2] \tag{34}$$

However, if we substitute from Eq. (14)

$$\omega = \frac{h - I(\theta)}{R^2(\theta)} \tag{35}$$

in Eq. (34), we obtain

$$A(\theta, \omega, \omega', \omega'') = \frac{R''}{R} - 2\left(\frac{R'}{R}\right)^2 + \frac{f(R)R^4}{[h - I(\theta)]^2} \tag{36}$$

It follows now from Eq. (17) that the right-hand side of Eq. (36) equals 1. As a result, Eqs. (31–33) take the simplified form

$$x'' = 2y' + 2\omega^{-1}u_1(\theta)y + T_1/mB(\theta) \tag{37a}$$

$$y'' = -2x' - \left\{[q(\theta) - \omega' u_1(\theta) - u_2(\theta)]\big/\omega^2\right\}y + T_2/mB(\theta) \tag{37b}$$

$$z'' = -z + T_3/mB(\theta) \tag{37c}$$

We observe that, when $|u_1(\theta)y/\omega| \ll |2y'|$ in Eq. (37a) and $\boldsymbol{T} = 0$, Eqs. (37a) and (37b) reduce to one nontrivial equation,

$$y'' + 4y + \left\{[q(\theta) - \omega' u_1(\theta) - u_2(\theta)]\big/\omega^2\right\}y = c$$

where c is an arbitrary constant.

Gravitational Field Case

In this section we specialize our considerations to a system consisting of a satellite and a spacecraft moving under the influence of the gravitational field of a central body and the linear drag force of the upper atmosphere.

Linear Drag in a Gravitational Field

The motion of a body under these forces was modeled by several authors[12–14] in the form

$$\ddot{\boldsymbol{R}} = -\mu \boldsymbol{R}/R^3 - \alpha \dot{\boldsymbol{R}}/R^2 \tag{38}$$

where μ and α are constants.

Following Leach[14] with a change of notation, and using Eqs. (14) and (15), the angular momentum of a body whose equation of motion is Eq. (38) is given by

$$L = R^2\dot{\theta} = h - \alpha\theta \tag{39}$$

where h is a constant. The orbit

$$R = (h^2/\mu)\{1/[J\cos\theta - g(\theta)]\} \tag{40}$$

satisfies Eq. (17) where [see Eq. (11)]

$$J = (h^2/\mu)|\boldsymbol{J}| \tag{41}$$

$$g(\theta) = k_3^2[\cos(\theta - h/\alpha)Ci(\theta - h/\alpha) + \sin(\theta - h/\alpha)Si(\theta - h/\alpha)] \tag{42}$$

and (see Ref. 23, p. 231)

$$Si(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t,\quad Ci(x) = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\,\mathrm{d}t$$

(γ is Euler's constant; see Ref. 23, p. 255). In these equations, $\boldsymbol{J}$ is the Runge–Lenz vector whose length is constant and $k_3 = h/\alpha$. From Eqs. (39) and (40), it now follows that

$$\omega = \dot{\theta} = k_2^2(\theta)[J\cos\theta - g(\theta)]^2 \tag{43}$$

where $k_2^2(\theta) = [(h - \alpha\theta)/h^4]\mu^2$. Moreover it is easy to verify directly that $g(\theta)$ satisfies the second-order differential equation

$$g''(\theta) + g(\theta) = -[h^2/(h - \alpha\theta)^2] \tag{44}$$

Following the same line of approximations and transformations that we carried in the last section for the motion of the spacecraft, we now obtain the equivalent form of Eqs. (25–27)

$$\omega x'' + \omega' x' = \left(\omega - k\omega^{\frac{1}{2}}\right)x + \omega(2y' - k_1 x') + (\omega' - k_1\omega)y + T_1/m\omega \tag{45}$$

$$\omega y'' + \omega' y' = \left(\omega + 2k\omega^{\frac{1}{2}} - k_1\omega' - k_1^2\omega\right)y - \omega(2x' + k_1 y') - (\omega' + k_1\omega)x + T_2/m\omega \tag{46}$$

$$\omega z'' + \omega' z' = -k_1\omega z' - k\omega^{\frac{1}{2}}z + T_3/m\omega \tag{47}$$

where

$$k(\theta) = \mu/(h - \alpha\theta)^{\frac{3}{2}}, \qquad k_1(\theta) = \alpha/(h - \alpha\theta) \tag{48}$$

Now introduce the transformation

$$(\bar{x}, \bar{y}, \bar{z}) = [\omega/(h - \alpha\theta)]^{\frac{1}{2}}(x, y, z) \tag{49}$$

After some algebra Eqs. (45–47) take the form

$$x'' + [A(\theta, \omega, \omega', \omega'') - 1]x - 2y' + 2k_1 y = T_1/mB_g(\theta) \tag{50}$$

$$y'' + \left\{[A(\theta, \omega, \omega', \omega'') - 1] - 3k\omega^{-\frac{1}{2}} + k_1(\omega'/\omega) + k_1^2\right\}y + 2x' = T_2/mB_g(\theta) \tag{51}$$

$$z'' + A(\theta, \omega, \omega', \omega'')z = T_3/mB_g(\theta) \tag{52}$$

where

$$B_g(\theta) = (h - \alpha\theta)^{\frac{1}{2}}\omega^{\frac{3}{2}} \tag{53}$$

$$A(\theta, \omega, \omega', \omega'') = -\tfrac{1}{2}(\omega''/\omega) + \tfrac{1}{4}(\omega'/\omega)^2 + k\omega^{-\frac{1}{2}} - \tfrac{3}{4}k_1^2 - \tfrac{1}{2}k_1(\omega'/\omega) \tag{54}$$

It is easy now to verify, using Eqs. (43) and (44) or from the general result in the preceding setting, that $A(\theta, \omega, \omega'\omega'') = 1$ and Eqs. (49–51) take the form

$$x'' - 2y' + [2\alpha/(h - \alpha\theta)]y = T_1/mB_g(\theta) \tag{55}$$

$$y'' + 2x' - \left[3k(\theta)\omega^{-\frac{1}{2}} - \alpha\omega'/\omega(h - \alpha\theta) - \alpha^2/(h - \alpha\theta)^2\right]y = T_2/mB_g(\theta) \tag{56}$$

$$z'' + z = T_3/mB_g(\theta) \tag{57}$$

State-Transition Matrix for a Special Case

An important special case that is amenable to analysis occurs if $\alpha \ll 1$ and $h \gg 1$. This analysis is based on the assumption that the contribution to the acceleration caused by drag is almost identical on both the satellite and the spacecraft. Because of the term $h - \alpha\theta$ that appears in the denominator, this simplified analysis is applicable only for time intervals of relatively short duration. In this special case the terms containing α in Eqs. (55) and (56) are negligible, and these equations can be approximated by

$$x'' - 2y' = T_1/mB_g(\theta) \tag{58}$$

$$y'' + 2x' - \left(\mu\big/h^{\frac{3}{2}}\right)\omega^{-\frac{1}{2}}y = T_2/mB_g(\theta) \tag{59}$$

When $\boldsymbol{T} = 0$, Eqs. (58) and (59) reduce, using Eq. (43), to one second-order equation

$$y'' + \{4 - 3/[J\cos\theta - g(\theta)]\}y = c \tag{60}$$

where c is a constant.

An exact solution to the homogeneous part of Eq. (60) is given by

$$\phi_1(\theta) = (\mu/h^2)[J\sin\theta + g'(\theta)][J\cos\theta - g(\theta)] \tag{61}$$

By the method of reduction of order, a complementary solution of the homogeneous equation is

$$\phi_2(\theta) = \phi_1(\theta)\int \phi_1(\theta)^{-2}\,\mathrm{d}\theta \tag{62}$$

Following the approach of Ref. 15, the complete solution of Eq. (60) is

$$y(\theta) = c_1\phi_1(\theta) + c_2\phi_2(\theta) + c_3\phi_3(\theta) \tag{63}$$

where

$$\phi_3(\theta) = 2\left[\phi_1(\theta)\int \phi_2(\theta)\,\mathrm{d}\theta - \phi_2(\theta)\int \phi_1(\theta)\,\mathrm{d}\theta\right] \tag{64}$$

Integrating the homogeneous form of Eq. (58), substituting Eq. (63), then integrating again reveals

$$x(\theta) = 2c_1\int \phi_1(\theta)\,\mathrm{d}\theta + 2c_2\int \phi_2(\theta)\,\mathrm{d}\theta + c_3\int [2\phi_3(\theta)+1]\,\mathrm{d}\theta + c_4 \tag{65}$$

The homogeneous form of Eq. (57) has solutions that are sinusoidal. The solution of the homogeneous system (58), (59), and (57) can be expressed through the fundamental matrix solution:

$$\Phi(\theta) = \begin{bmatrix} 2\int \phi_1(\theta)\,\mathrm{d}\theta & 2\int \phi_2(\theta)\,\mathrm{d}\theta & \int [2\phi_3(\theta)+1]\,\mathrm{d}\theta & 1 & 0 & 0 \\ 2\phi_1(\theta) & 2\phi_2(\theta) & 2\phi_3(\theta)+1 & 0 & 0 & 0 \\ \phi_1(\theta) & \phi_2(\theta) & \phi_3(\theta) & 0 & 0 & 0 \\ \phi_1'(\theta) & \phi_2'(\theta) & \phi_3'(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{66}$$

When $\boldsymbol{c} = (c_1, c_2, c_3, c_4, c_5, c_6)^T$, the state vector is represented by

$$[x(\theta), x'(\theta), y(\theta), y'(\theta), z(\theta), z'(\theta)]^T = \Phi(\theta)\boldsymbol{c} \tag{67}$$

The fundamental matrix solution $\Phi(\theta)$ may be inverted by finding a fundamental matrix solution for the adjoint system as outlined in Ref. 15. The result is

$$\Phi^{-1}(\theta) = \begin{bmatrix} 0 & -2\int \phi_2(\theta)\,\mathrm{d}\theta & 4\int \phi_2(\theta)\,\mathrm{d}\theta + \phi_2'(\theta) & -\phi_2(\theta) & 0 & 0 \\ 0 & 2\int \phi_2(\theta)\,\mathrm{d}\theta & -4\int \phi_1(\theta)\,\mathrm{d}\theta - \phi_1'(\theta) & \phi_1(\theta) & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 1 & -\int [2\phi_3(\theta)+1]\,\mathrm{d}\theta & 2\int [2\phi_3(\theta)+1]\,\mathrm{d}\theta + \phi_3'(\theta) & -\phi_3(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{68}$$

This establishes a state-transition matrix

$$M(\theta, \theta_0) = \Phi(\theta)\Phi(\theta_0)^{-1} \tag{69}$$

that applies in this special case.

Various Types of Control Models

In this section we consider various control models associated with the rendezvous problems discussed heretofore. In these models, various performance indices and constraints on the spacecraft thrusters can be assumed. In these models, the dynamics are linear and expressed in the general form

$$\hat{\boldsymbol{x}}'(\theta) = \hat{A}(\theta)\hat{\boldsymbol{x}}(\theta) + \hat{B}(\theta)\boldsymbol{T}(\theta) \tag{70}$$

where the symbols $\hat{A}$ and $\hat{B}$ do not refer to the symbols A and B of preceding sections.

In all of these models, $\hat{A}(\theta)$ and $\hat{B}(\theta)$ assume a form that generalizes that of Ref. 15. In these models,

$$\hat{A}(\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta(\theta) & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & \eta(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{71}$$

$$\hat{B}(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1/mB(\theta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/mB(\theta) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/mB(\theta) \end{bmatrix} \tag{72}$$

These reduce to the same form as the rendezvous equations in a central force field without drag[15] if $\zeta(\theta)$ is identically zero and $mB(\theta)$ is identically unity. The results for a general central force of the present paper, that is, Eqs. (37), are represented by

$$\zeta(\theta) = -\omega^{-1}u_1(\theta) \tag{73a}$$

$$\eta(\theta) = [-q(\theta) + \omega' u_1(\theta) + u_2(\theta)]\big/\omega^2 \tag{73b}$$

The more specialized considerations for the gravitational field, that is, Eqs. (55–57), are represented by

$$\zeta(\theta) = -2\alpha/(h - \alpha\theta) \tag{74a}$$

$$\eta(\theta) = -3k(\theta)\omega^{-\frac{1}{2}} + \alpha\omega'/\omega(h - \alpha\theta) + \alpha^2/(h - \alpha\theta)^2 \tag{74b}$$

If one uses the approximate Eqs. (58) and (59), these expressions simplify to

$$\zeta(\theta) = 0 \tag{75a}$$

$$\eta(\theta) = G[\omega(\theta)] = \left(\mu/h^{\frac{3}{2}}\right)\omega(\theta)^{-\frac{1}{2}} \tag{75b}$$

where $G(\omega)$ is the negative of the coefficient of y in Eq. (59) and follows the notation of Ref. 15. In this important special case, the complete state-transition matrix is obtained from Eqs. (66)–(69).

Constant Exhaust Velocity and Power-Limited Models

In the model of minimum-fuel rendezvous with constant exhaust velocity,[8] one seeks to minimize the cost function,

$$J[\boldsymbol{T}] = \int_{\theta_0}^{\theta_f} \gamma(\theta)|\boldsymbol{T}(\theta)|\,\mathrm{d}\theta \tag{76}$$

defined over the flight interval $\theta_0 \le \theta \le \theta_f$, where $|\boldsymbol{T}(\theta)| = [\boldsymbol{T}(\theta)\cdot\boldsymbol{T}(\theta)]^{1/2}$ and γ is a weighting function, subject to a positive bound T_b on the magnitude of the thrust

$$|\boldsymbol{T}(\theta)| \le T_b \tag{77}$$

Additionally Eqs. (70) and the boundary conditions

$$\hat{\boldsymbol{x}}(\theta_0) = \boldsymbol{x}_0, \qquad \hat{\boldsymbol{x}}(\theta_f) = \boldsymbol{x}_f \tag{78}$$

must be satisfied. This problem has been studied, necessary and sufficient conditions formulated for its solution,[16] and numerical solutions of two-point boundary-value problems were calculated for certain examples.[17]

Another model is that of power-limited rendezvous with variable exhaust velocity.[18] In this model the cost function (76) is replaced by the function

$$J[\boldsymbol{T}] = \int_{\theta_0}^{\theta_r} \gamma(\theta)|\boldsymbol{T}(\theta)|^2\,\mathrm{d}\theta \tag{79}$$

and the magnitude of the thrust is unbounded. The state equations (70) and the boundary conditions (78) are the same. Under appropriate conditions, the form of the optimal thrust function is well known and is presented in previous work.[18]

Another variation is that of power-limited rendezvous with a bound on the magnitude of the thrust. In this example, the function (79) is to be minimized subject to the constraint (77). This variation has also been studied, and necessary and sufficient conditions for its solution have been found.[19]

Some applications of electric propulsion systems indicate that, in the case of power-limited rendezvous, there are both upper and lower bounds on thrust magnitude:

$$T_{b_1} \le |\boldsymbol{T}(\theta)| \le T_{b_2} \tag{80}$$

This problem also has been studied and numerical solution of certain two-point boundary-value solutions has been attained.[20] In the case of multiple thrusters, constraint (80) generalizes to a set of shells of multiple thrust in power-limited rendezvous. This problem also has been studied[21] and numerical simulations performed.

Impulsive Models and State-Transition Matrices

The impulsive rendezvous problem based on the linearized form (70) has been analyzed, and necessary and sufficient conditions for its solution have been formulated.[22]

The impulsive problem[22] is to find velocity increments $\Delta\boldsymbol{v}_1, \ldots, \Delta\boldsymbol{v}_k \in \Re^3$ and their points of application $\theta_1, \ldots, \theta_k \epsilon [\theta_0, \theta_f]$ to minimize the total weighted characteristic velocity

$$J(\Delta\boldsymbol{v}_1, \ldots, \Delta\boldsymbol{v}_k, \theta_1, \ldots, \theta_k) = \sum_{i=1}^{k} \gamma(\theta_i)|\Delta\boldsymbol{v}_i| \tag{81}$$

subject to the constraint

$$\sum_{i=1}^{k} \Phi(\theta_i)^{-1}\hat{B}(\theta_i)\Delta\boldsymbol{v}_i = \Phi(\theta_f)^{-1}\hat{\boldsymbol{x}}_f - \Phi(\theta_0)^{-1}\hat{\boldsymbol{x}}_0 \tag{82}$$

where $\Phi(\theta)$ is any fundamental matrix solution associated with $\hat{A}(\theta)$.

The fundamental matrix solution $\Phi(\theta)$ is of paramount importance in impulsive problems. A transfer from any state $\hat{\boldsymbol{x}}(\theta_1)$ to any other state $\hat{\boldsymbol{x}}(\theta_2)$ can be accomplished by the multiplication of $\hat{\boldsymbol{x}}(\theta_1)$ by the state transition matrix $\Phi(\theta_2)\Phi(\theta_1)^{-1}$ if no impulse occurs on the interval $(\theta_1 < \theta < \theta_2)$.

The structure of the solution of the problem of optimal rendezvous in a general central force field[11] allows one to find a remarkably simple form for a state transition matrix in this case.[15] When drag is introduced into the equations using the approximations (75), this simplicity is preserved, and the state transition matrix is calculated from Eq. (69). The problem becomes more complex if the matrix $\hat{A}(\theta)$ is defined through Eqs. (73) or (74). These additional complexities may require a numerical solution.

Conclusions

A set of rendezvous problems was formulated for a central-force field in the presence of linear drag. For these problems, some mathematical transformations show that the motion can be described by surprisingly simple sets of equations. Under the simplifying assumptions of equal drag accelerations on the satellite and the spacecraft, the representative differential equations can be integrated to construct a state-transition matrix having the identical structure found in previous work in which drag was not considered. The work is then placed in the context of previous results on bounded-thrust, power-limited, and impulsive rendezvous problems and can be used for the construction of state-transition matrices.

Although a linear drag model may be too crude for practical application, the authors believe the approach presented herein will be of benefit in future analytical studies. Similar results using a more practical quadratic drag model are expected in the future.

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